Fundamental Regions for Certain Finite Groups in S_4.

BY HENRY F. PRICE.

One of the most interesting results of the study of transformations is what Klein has termed the "fundamental region."

A fundamental region for a group of transformations is a system of points which contains one and only one point of every conjugate set.

Fundamental regions in the complex plane have been studied for some time and are well known. Klein * and his followers have developed the subject to a considerable extent. There is a close relationship between this subject and the elliptic modular functions and the reduction of quadratic forms.

The fundamental regions for groups in more than one complex variable have not been studied much. However, J. W. Young,† in a recent paper, obtained such regions for cyclic groups in two complex variables.

In this paper will be considered fundamental regions for certain finite groups in two complex variables. The octahedral and icosahedral groups will be dealt with.

The fundamental regions for these groups in the real plane can be readily determined and found to be triangles bounded by the axes of reflections. In the case of the complex plane the problem is solved by using Hermitian forms which meet the real plane in the sides of these triangles.

The problem will be solved completely in the case of the octahedral group. In the case of the icosahedral group it will be solved except for the points which reduce one or more of the Hermitian forms to zero.

The ternary collineation group G_{24} can be generated by the following three operations: $E_1[-\xi_1, \xi_3, \xi_2]$, $E_2[\xi_2, \xi_1, \xi_3]$ and $E_3[\xi_1, \xi_3, \xi_2]$.

It permutes the points of the real plane. As it contains nine operations of order 2, there are nine reflections. As the group is simply isomorphic with the symmetric group on four letters, it is evident that these reflections are in

* Felix Klein, "Elliptischen Modulfunktionen, Vol. I, pp. 183–207.

† J. W. Young, "Fundamental Regions for Cyclical Groups of Linear Fractional Transformations on Two Complex Variables," *Bull. Amer. Math. Soc.*, Vol. XVII, p. 340.

two conjugate sets. The axes of the reflections, $\xi_1 = \pm\xi_3$, $\xi_2 = \pm\xi_3$, $\xi_1 = \pm\xi_2$, $\xi_1 = 0$, $\xi_2 = 0$ and $\xi_3 = 0$, divide the plane into twenty-four triangles.

Any point in one of these triangles can be transformed, by a suitably chosen operation of the group, into a point in any other triangle. Any triangle is then a *fundamental region* in the plane for the group G_{24}.

The group permutes the complex points of the plane also. If we consider the totality of points in the plane, complex as well as real, as real points in four-space, we may ask the question whether fundamental regions exist in S_4 for the group under consideration. If one does exist it must contain one triangle, and only one, of the real plane. The fixed points of the transformations would lie on the boundaries of such a fundamental region.

Consider the Hermitian forms $\xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2$ and $\xi_1\bar{\xi}_2 + \bar{\xi}_1\xi_2$ in which $\frac{\xi_1}{\xi_3} = x + iu$ and $\frac{\xi_2}{\xi_3} = y + iv$. Under the G_{24} we have two conjugate sets of three forms each:

$$
\begin{array}{llcll}
(1) & \xi_1\bar{\xi}_1 - \xi_2\bar{\xi}_2, & & (4) & \xi_1\bar{\xi}_2 + \bar{\xi}_1\xi_2, \\
(2) & \xi_2\bar{\xi}_2 - \xi_3\bar{\xi}_3, & \text{and} & (5) & \xi_2\bar{\xi}_3 + \bar{\xi}_2\xi_3, \\
(3) & \xi_3\bar{\xi}_3 - \xi_1\bar{\xi}_1, & & (6) & \xi_3\bar{\xi}_1 + \bar{\xi}_3\xi_1.
\end{array}
$$

It is evident that there is at least one relation between the forms, *i. e.*, $(1) + (2) + (3) \equiv 0$.

If we consider the portion of S_4 in which the signs of (1), (3), (4), (5) and (6) are all $+$ and make use of the relation $(1) + (2) + (3) \equiv 0$, we see that the sign of (2) is determined as $-$.

This region will be written $[+-+, +++]$ where the signs of the six forms are written in order.

We shall next consider into how many such regions S_4 is divided by the six Hermitian forms.

The forms (1), (2) and (3) are conjugate under G_{24}, and because of the relation $(1) + (2) + (3) \equiv 0$ admit at most six arrangements of sign. The forms (4), (5) and (6) are also conjugate under G_{24}, and admit at most eight arrangements of sign. There are therefore forty-eight possible choices of sign for (1), (2) .... (6). But the eight arrangements divide into two complete conjugate sets of four under G_{24} according as the number of $+$ signs is odd or even. For one of such four choices for the forms (4), (5), (6), the six arrangements of sign for (1), (2), (3) all are conjugate, *e. g.*, the $+++$ choice is unaltered by the group G_6 of permutations of the variables, and the G_6 permutes all arrangements for (1), (2), (3).

Hence the forty-eight possible choices of sign or possible regions in S_4 divide into two conjugate sets of twenty-four each, and two of these regions, one from each set, constitute a fundamental region in S_4 for G_{24}.

The region $[+--+, +++]$ belongs to one of the sets. By changing the sign of (5) to $-$ we obtain a region $[+--+, +--+]$ of the other set.

Taking these two regions together we obtain $\Gamma = [+--+, +\pm+]$ which is a fundamental region in S_4 for the group G_{24}, except for the points which reduce one or more of the Hermitian forms to zero.

If we consider the points which reduce one or more of the forms to zero we are dealing with what we may call the "boundaries" of the fundamental regions.

By placing the six forms in Γ equal to zero singly, in pairs, in groups of three, etc., in all possible ways, and discarding those which are conjugates of others, it is found that there are twenty-three sets of points which are sections of Γ's boundaries and which should be taken in the fundamental region for the group. Γ can be defined completely, therefore, by the sets of points:

$$
\begin{array}{cccc}
 & & [+--+,\ +\pm+], & \\
[+--+,\ +0+], & [0-+,\ +\pm+], & [+-0,\ +\pm+], & [+--+,\ 0++], \\
[+--+,\ ++0], & [0-+,\ +0+], & [+-0,\ +0+], & [+--+,\ 000], \\
[+--+,\ 0+0], & [0-+,\ 0++], & [+-0,\ ++0], & [000,\ +\pm+], \\
[+--+,\ 00+], & [0-+,\ 00+], & [+-0,\ 00+], & [000,\ +0+], \\
[+--+,\ +00], & [0-+,\ +00], & [+-0,\ 0+0], & [000,\ 00+], \\
 & [0-+,\ 000], & [+-0,\ 000]. &
\end{array}
$$

The ternary collineation group G_{60} furnishes a more complex fundamental region in S_4 than G_{24} does. It is well known that this group can be generated by the three operations:

$$E_1[\xi_2, \xi_3, \xi_1]; \quad E_2[\xi_1, -\xi_2, -\xi_3];$$

and

$$E_3 \begin{cases} \xi_1' = \xi_1 - \alpha\xi_2 + (\alpha+1)\xi_3, \\ \xi_2' = -\alpha\xi_1 + (\alpha+1)\xi_2 + \xi_3, \\ \xi_3' = (\alpha+1)\xi_1 + \xi_2 - \alpha\xi_3, \end{cases}$$

where $\alpha = \dfrac{-1 \pm \sqrt{5}}{2}$.*

This group permutes the points of the real plane. As it contains fifteen operations of order 2, there are fifteen axes of reflections. These lines divide

* H. H. Mitchell, "Determination of the Ordinary and Modular Ternary Linear Groups," *Trans. Amer. Math. Soc.*, Vol. XII, No. 2, p. 223.

the plane into sixty triangles. The intersections of these fixed lines are *real* fixed points of three classes; first, the points left invariant under the fifteen subgroups of order 4; second, the points invariant under the ten subgroups of order 6; and third, the points invariant under the six subgroups of order 10.

Each of the sixty triangles into which the plane is divided by the fifteen fixed lines has for its vertices one of each of the three classes of fixed points.

Each of the sixty triangles is a fundamental region in the plane. The group also permutes the complex points of the plane. Just as in the case of the G_{24} we can consider the totality of real and complex points in the plane as real points in S_4 and seek a fundamental region for the group G_{60} in the higher space.

Consider the Hermitian form $2\xi_1\bar{\xi}_2+2\bar{\xi}_1\xi_2$. Under G_{60} there is a single set of fifteen forms conjugate to $2\xi_1\bar{\xi}_2+2\bar{\xi}_1\xi_2$, which can be expressed in terms of six forms:

$$
\begin{aligned}
F_1 &= (\alpha-1)\xi_1\bar{\xi}_1+(\alpha+2)\xi_2\bar{\xi}_2-(2\alpha+1)\xi_3\bar{\xi}_3+3\xi_1\bar{\xi}_2+3\bar{\xi}_1\xi_2,\\
F_2 &= (\alpha+2)\xi_1\bar{\xi}_1-(2\alpha+1)\xi_2\bar{\xi}_2+(\alpha-1)\xi_3\bar{\xi}_3-3\xi_3\bar{\xi}_1-3\bar{\xi}_3\xi_1,\\
F_3 &= -(2\alpha+1)\xi_1\bar{\xi}_1+(\alpha-1)\xi_2\bar{\xi}_2+(\alpha+2)\xi_3\bar{\xi}_3+3\xi_2\bar{\xi}_3+3\bar{\xi}_2\xi_3,\\
F_4 &= -(2\alpha+1)\xi_1\bar{\xi}_1+(\alpha-1)\xi_2\bar{\xi}_2+(\alpha+2)\xi_3\bar{\xi}_3-3\xi_2\bar{\xi}_3-3\bar{\xi}_2\xi_3,\\
F_5 &= (\alpha+2)\xi_1\bar{\xi}_1-(2\alpha+1)\xi_2\bar{\xi}_2+(\alpha-1)\xi_3\bar{\xi}_3+3\xi_3\bar{\xi}_1+3\bar{\xi}_3\xi_1,\\
F_6 &= (\alpha-1)\xi_1\bar{\xi}_1+(\alpha+2)\xi_2\bar{\xi}_2-(2\alpha+1)\xi_3\bar{\xi}_3-3\xi_1\bar{\xi}_2-3\bar{\xi}_1\xi_2.
\end{aligned}
$$

The fifteen forms conjugate to $2\xi_1\bar{\xi}_2+2\bar{\xi}_1\xi_2$ are $F_{ik}=-F_{ki}=F_i-F_k$. There are twenty relations between these forms $F_{ij}+F_{jk}+F_{ki}\equiv 0$.

It is found that the Hermitian forms F_{12}, F_{54}, F_{16}, F_{34} and F_{52} meet the real plane in the sides of the triangle whose sides are $x+\alpha y-\alpha^2=0$, $x=0$ and $y=0$.

For any point which lies in this triangle, the signs of F_{12} and F_{54} are both $-$, while those of F_{16}, F_{34} and F_{52} are all $+$.

Taking F_{12} and F_{54} as negative and the other three forms as positive, and making use of the twenty relations between the forms, it is seen that the signs of the remaining ten forms are determined. Therefore, none of these ten forms can *cross* the *region* in S_4 which is determined by the five forms under consideration. Since for this region $F_3>F_4>F_5>F_2>F_1>F_6$, it can be written [345216].

Under G_{60} the region [345216] has sixty conjugate regions. These meet the real plane in distinct triangles, the fundamental regions, for the group, in the plane.

The question arises into how many such regions is S_4 divided by the forms F_{ik}?

For any point in S_4 not on an F_{ik} the values of the F_i are all distinct, and, since these six values admit at most seven hundred and twenty permutations, the forms F_{ik} have at most seven hundred and twenty arrangements of sign.

Under G_{60} the seven hundred and twenty value systems of the F_i divide into twelve conjugate sets of sixty each, and if one value system is taken from each set we obtain a fundamental region. As an example of such a fundamental region we give the twelve value systems determined by the inequalities $F_3 > F_4 > F_1$, $F_4 > F_6$, $F_5 > F_2 > F_1$ and $F_2 > F_6$. No two of these are conjugates under G_{60} and therefore they comprise a fundamental region in S_4 for the group, except for the points which reduce one or more of the F_{ik} to zero.